On integrability of infinitesimal actions

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Abstract

We use foliations and connections on principal Lie groupoid bundles to prove various integrability results for Lie algebroids. In particular, we show, under quite general assumptions, that the semi-direct product associated to an infinitesimal action of one integrable Lie algebroid on another is integrable. This generalizes recent results of Dazord and Nistor.

Introduction

Lie algebroids have recently turned out to be very useful in several ways, e.g. related to deformation quantization of manifolds and to Poisson geometry (see for example [4, 5, 11, 12, 21]). The notion of a Lie algebroid itself goes back to Pradines [24, 25]. He constructed for each Lie groupoid such a Lie algebroid, and outlined a Lie theory describing a correspondence between Lie groupoids and algebroids completely analogous to the classical theory for Lie groups and algebras. It remained a problem to develop the details of this theory, until Almeida and Molino [1] provided a counterexample to one of Pradines main assertions: they proved that a transversely complete foliation gives rise to a transitive Lie algebroid which is integrable if and only if the foliation is developable. Nevertheless, there are some well-known classical examples of Lie algebroids which can be integrated. For example, Douady and Lazard [8] proved that any bundle of Lie algebras can be integrated to a bundle of Lie groups, which in the language of Lie algebroids means that any Lie algebroid with trivial anchor map is integrable.

Recently, several positive integrability results have been discovered. Dazord [6] proved that the transformation Lie algebroid associated to an infinitesimal action of a Lie algebra on a manifold is integrable to a Lie groupoid. Furthermore, using the integrability of foliation algebroids and the result of Douady and Lazard, Nistor [19] proved that any regular Lie algebroid which admits a flat splitting is integrable. His motivation for this result was to construct examples of pseudodifferential operators on groupoids [20]. In a recent thesis, Debord

proved that any Lie algebroid with almost injective anchor map is integrable [7].

The purpose of this paper is to prove some integrability results, which include the ones of Dazord and Nistor. More specifically, our main results concern actions of one Lie algebroid on another. We prove that in many cases, the semi-direct product of such an infinitesimal action (described in [10]) is integrable whenever each of the algebroids is.

The outline of this paper is as follows. In the first section, we recall some basic definitions and main examples concerning Lie algebroids and Lie groupoids, and fix the notations. In the second section, we discuss principal G-bundles for a Lie groupoid G, and introduce a notion of connection which takes values in the Lie algebroid of G. In the third section, we first give a quick and uniform treatment of some basic results of the Lie theory for groupoids. More specifically, we construct for each Lie groupoid a source-simply connected cover having the same algebroid (the source-simply connected Lie groupoids play the role analogous to the simply connected Lie groups). The existence of such a cover was proved earlier by more involved methods in some special cases, e.g. [3, 13]; see [14] for a survey. Furthermore, we use foliation theory and connections to show that integrability is inherited by subalgebroids, and to give a simple proof of Mackenzie and Xu's result concerning integrability of morphisms between algebroids [17]. In this section, we also prove an integrability theorem stating that an action of a Lie groupoid G on an integrable Lie algebroid $\mathfrak{h}$ can be integrated to an action of G on the integral groupoid of $\mathfrak{h}$.

In section 4, we discuss derivations on Lie algebroids. For the Lie algebroid $\mathfrak{h}$ of a source-simply connected groupoid H, we prove that the Lie algebra $\mathrm{Der}(\mathfrak{h})$ of derivations on $\mathfrak{h}$ is isomorphic to the Lie algebra of multiplicative vector fields on H. We also discuss actions of another Lie algebroid $\mathfrak{g}$ on $\mathfrak{h}$ in terms of the algebra $\mathrm{Der}(\mathfrak{h})$, and recall the construction of semi-direct products from [10].

The main results of this paper are contained in section 5. Here we prove that under quite general assumptions, the semi-direct product of integrable Lie algebroids is itself integrable. More specifically, this holds if the algebroid $\mathfrak{h}$ is a foliation (Theorem 5.1), or if $\mathfrak{h}$ is "proper" over $\mathfrak{g}$ in a suitable sense (Corollary 5.4), or if $\mathfrak{h}$ is integrable by a source-compact source-simply connected Lie groupoid (Theorem 5.7). At present, we do not know whether in general $\mathfrak{g} \ltimes \mathfrak{h}$ is integrable if $\mathfrak{g}$ and $\mathfrak{h}$ are.

1 Lie algebroids and Lie groupoids

1.1 Lie algebroids

In this section we recall the definition of a Lie algebroid as well as some of the main examples. For a more extensive discussion we refer to [4, 13]. Throughout this paper, we shall work in the smooth context, so "manifold" means smooth

manifold, "map" means smooth map, "vector bundle" means smooth real vector bundle, etc.

Let M be a manifold. A *Lie algebroid* over M is a smooth vector bundle $\pi : \mathfrak{g} \to M$, together with a map $\rho : \mathfrak{g} \to T(M)$ of vector bundles over M and a (real) Lie algebra structure $[\,\text{-}\,,\,\text{-}\,]$ on the vector space $\Gamma\mathfrak{g}$ of sections of $\mathfrak{g}$ such that

(i) the induced map $\Gamma(\rho) : \Gamma\mathfrak{g} \to \mathfrak{X}(M)$ is a Lie algebra homomorphism, and

(ii) the Leibniz identity

$$[X, fX'] = f[X, X'] + \Gamma(\rho)(X)(f)X'$$

holds for any $X, X' \in \Gamma\mathfrak{g}$ and any $f \in C^\infty(M)$.

The map ρ is called the *anchor* of the Lie algebroid $\mathfrak{g}$. The map $\Gamma(\rho)$ is often denoted by ρ as well, and also called the anchor. The manifold M is called the *base manifold* of the Lie algebroid $\mathfrak{g}$.

Examples 1.1 (i) Every finite dimensional Lie algebra is a Lie algebroid over a one point space.

(ii) Any manifold M can be viewed as a Lie algebroid in two ways, by taking the zero bundle over M (which we shall denote simply by M), or by taking the tangent bundle over M with the identity map for the anchor (we shall denote this Lie algebroid by $T(M)$).

(iii) Any vector bundle E over M can be viewed as a Lie algebroid over M, with zero bracket and anchor.

(iv) A vector bundle E over M with a smoothly varying Lie algebra structure on its fibers (i.e. a bundle of Lie algebras) can be viewed as a Lie algebroid over M with zero anchor.

(v) A foliation $\mathcal{F}$ of M is by definition an involutive (hence integrable) subbundle of $T(M)$. Thus a foliation of M is the same thing as a Lie algebroid over M with injective anchor map.

(vi) Let M be a manifold equipped with an infinitesimal action of a Lie algebra $\mathfrak{g}$, i.e. a Lie algebra homomorphism $\gamma : \mathfrak{g} \to \mathfrak{X}(M)$. The trivial bundle $\mathfrak{g} \times M$ over M has the structure of a Lie algebroid, with anchor given by $\rho(\xi, x) = \gamma(\xi)_x$, and Lie bracket

$$[u, v](x) = [u(x), v(x)] + (\gamma(u(x))(v))(x) - (\gamma(v(x))(u))(x) \ ,$$

for $u, v \in C^\infty(M, \mathfrak{g}) \cong \Gamma(M, \mathfrak{g} \times M)$ and $x \in M$. This Lie algebroid is called the *transformation* algebroid associated to the infinitesimal action.

(vii) Let (M, Π) be a Poisson manifold. Then there is a natural Lie algebra structure on $\Omega^1(M)$ which makes $T^*(M)$ into a Lie algebroid over M. The anchor of this algebroid is $-\tilde{\Pi}$, where $\tilde{\Pi} : T^*(M) \to T(M)$ is induced by the bivector field Π. For details, see e.g. [4].

Let $\mathfrak{g}$ be a Lie algebroid over M and $\phi : N \to M$ a map of manifolds. Consider the pull-back bundle $\phi^*\mathfrak{g}$. The sections of the form $\phi^*(X) = (id, X \circ \phi)$, $X \in \Gamma\mathfrak{g}$, span $\Gamma\phi^*\mathfrak{g}$ as a $C^\infty(N)$-module. In fact, the map $C^\infty(N) \otimes_{C^\infty(M)} \Gamma\mathfrak{g} \to$

$\Gamma\phi^*\mathfrak{g}$, which sends $f \otimes X$ to $f\phi^*(X)$, is an isomorphism [9].

$$\phi^*\mathfrak{g} = N \times_M \mathfrak{g} \longrightarrow \mathfrak{g}$$

Let $\mathfrak{h}$ be a Lie algebroid over N, and $\Phi : \mathfrak{h} \to \mathfrak{g}$ a bundle map over $\phi : N \to M$. A Φ-*decomposition* of a section $Y \in \Gamma\mathfrak{h}$ is a decomposition $(id, \Phi \circ Y) = \sum_i f_i \phi^*(X_i) \in \Gamma\phi^*\mathfrak{g}$, for some $f_i \in C^\infty(N)$ and $X_i \in \Gamma\mathfrak{g}$. Such a bundle map $\Phi : \mathfrak{h} \to \mathfrak{g}$ over ϕ is a *morphism* of Lie algebroids [10] if it preserves the anchor, i.e. $\rho \circ \Phi = d\phi \circ \rho$, and if it preserves the bracket in the following sense: for any $Y \in \Gamma\mathfrak{h}$ with a Φ-decomposition $\sum_i f_i \phi^*(X_i)$, and any $Y' \in \Gamma\mathfrak{h}$ with a Φ-decomposition $\sum_j f'_j \phi^*(X'_j)$,

$$\sum_{i,j} f_i f'_j \phi^*([X_i, X'_j]) + \sum_j \rho(Y)(f'_j)\phi^*(X'_j) - \sum_i \rho(Y')(f_i)\phi^*(X_i)$$

is a Φ-decomposition of $[Y, Y']$.

1.2 Lie groupoids

Like the previous one, this section only serves to recall some basic definitions and fix the notations.

A groupoid is a small category G in which all the arrows are invertible. We shall write G_0 for the set of objects of G, while the set of arrows of G will be denoted by G_1. We shall often identify G_0 with the subset of units of G_1. The structure maps of G will be denoted as follows: $\alpha, \beta : G_1 \to G_0$ will stand for the source (domain) map, respectively the target (codomain) map, $\mu : G_1 \times_{G_0} G_1 \to G_1$ ($\mu(g, g') = gg'$) for the multiplication (composition) map, $inv : G_1 \to G_1$ ($inv(g) = g^{-1}$) for the inverse map and $uni : G_0 \to G_1$ ($uni(x) = 1_x$) for the unit map. We sometimes say that G is a groupoid over G_0.

A *Lie groupoid* is a groupoid G, equipped with the structure of smooth manifold both on the set of arrows G_1 and on the set objects G_0, such that all the structure maps of G are smooth and α is a submersion. Note that this implies that β is a submersion as well, that there is a natural smooth structure on the domain $G_1 \times_{G_0} G_1$ of the multiplication, and that uni is an embedding. We shall assume that the manifold of objects G_0 and the α-fibers $\alpha^{-1}(x)$, $x \in G_0$, are Hausdorff, but we do not assume G_1 to be Hausdorff (cf. Example 1.2 (v)).

A morphism of Lie groupoids $F : H \to G$ is a functor which is smooth as a map between the manifolds of arrows ($F_1 : H_1 \to G_1$) and as a map between the manifolds of objects ($F_0 : H_0 \to G_0$).

Examples 1.2 (i) Every Lie group is a Lie groupoid over a one point space.

(ii) Any submersion $M \to B$ gives a Lie groupoid $M \times_B M$ over M, with $\alpha = pr_2$ and $\beta = pr_1$. In particular, to any manifold M we associate two Lie groupoids: the one corresponding to the identity $M \to M$, and the one (the pair groupoid) corresponding to the map from M to a space with only one point.

(iii) A Lie groupoid G with $\alpha = \beta$ is just a family of Lie groups smoothly parametrized by G_0. In particular, any vector bundle is a Lie groupoid with $\alpha = \beta$.

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(iv) The fundamental groupoid of a manifold is a Lie groupoid.

(v) A foliated manifold $(M, \mathcal{F})$ gives rise to two Lie groupoids, the holonomy groupoid $\mathrm{Hol}(M, \mathcal{F})$ and the monodromy (or homotopy) groupoid $\mathrm{Mon}(M, \mathcal{F})$. The manifold of objects is M in both cases. If $x, y \in M$ are on different leaves, there are no arrows from x to y in both cases. If x and y are on the same leaf L, the arrows from x to y in $\mathrm{Mon}(M, \mathcal{F})$ are homotopy classes of paths from x to y inside L. Thus $\mathrm{Mon}(M, \mathcal{F})$ is the union of the fundamental groupoids of the leaves, equipped with a suitable smooth structure. The holonomy groupoid is a quotient of the monodromy groupoid: for $x, y \in L$, the arrows from x to y in $\mathrm{Hol}(M, \mathcal{F})$ are the holonomy classes of paths from x to y inside L. These monodromy and holonomy groupoids are generally non-Hausdorff. For details, see [23, 28].

(vi) If M is a manifold equipped with a smooth left action of a Lie group G, the translation groupoid $G \ltimes M$ has M for its manifold of objects and $G \times M$ for its manifold of arrows, an arrow from x to y being a pair (g, x) with $gx = y$. The multiplication in $G \ltimes M$ is defined by $(g, x)(g', x') = (gg', x')$ when $x = g'x'$.

(vii) Let G be a Lie group and let $P \to M$ be a right principal G-bundle over M. The pair groupoid $P \times P$ over P of Example 1.2 (ii) has a natural diagonal G-action, and the quotient $(P \times P)/G$ is a Lie groupoid over M, called the *gauge* groupoid of the principal bundle.

1.3   The Lie algebroid of a Lie groupoid

The construction of a Lie algebra $\mathfrak{g}$ of a given Lie group G extends to groupoids [24, 13]. Explicitly, if G is a Lie groupoid, the vector bundle $T^{\alpha}(G_1) = \mathrm{Ker}(d\alpha)$ over G_1 of α-vertical tangent vectors pulls back along $uni : G_0 \to G_1$ to a vector bundle $\mathfrak{g}$ over G_0. This vector bundle has the structure of a Lie algebroid. Its anchor $\rho : \mathfrak{g} \to T(G_0)$ is induced by the differential of the target map, $d\beta : T(G_1) \to T(G_0)$. To define the bracket, note that the bundle $T^{\alpha}(G_1)$ over G_1 has a right G-action, and that the G-invariant sections of $T^{\alpha}(G_1)$ over G_1 form a Lie subalgebra of $\mathfrak{X}(G_1)$, which we denote by $\mathfrak{X}^{\alpha}_{inv}(G)$. Now the sections of $\mathfrak{g}$ over G_0 can be identified with the G-invariant sections of $T^{\alpha}(G_1)$ over G_1. Explicitly, a section X of $\mathfrak{g}$ gives an invariant α-vertical vector field on G_1 with value

$$X_{\beta(g)} g$$

at an arrow $g \in G_1$. This identification gives us a Lie algebra structure on $\Gamma\mathfrak{g}$; see [13] for details. We denote the Lie algebroids associated to G, H, etc. by $\mathfrak{g}$, $\mathfrak{h}$, etc, or sometimes by $\mathcal{A}(G)$, $\mathcal{A}(H)$, etc. The differential of a morphism $F : H \to G$ of Lie groupoids induces a morphism $\mathcal{A}(F) : \mathcal{A}(H) \to \mathcal{A}(G)$ of Lie algebroids over $F_0 : H_0 \to G_0$, in a functorial way [10].

Definition 1.3 *A Lie algebroid $\mathfrak{g}$ is called integrable if it is isomorphic to the Lie algebroid associated to a Lie groupoid G. If this is the case, then G is called an integral of $\mathfrak{g}$.*

REMARK. Contrary to the case of finite dimensional Lie algebras and Lie groups, there exist Lie algebroids which are not integrable. See [1, 18] for an example.

Examples 1.4 (i) Any finite dimensional Lie algebra is an integrable Lie algebroid ("Lie's third theorem").

(ii) Any foliation $\mathcal{F}$ of a manifold M is an integrable Lie algebroid, because we have $\mathcal{F} \cong \mathcal{A}(\mathrm{Hol}(M, \mathcal{F})) \cong \mathcal{A}(\mathrm{Mon}(M, \mathcal{F}))$.

(iii) Any bundle E of Lie algebras over M is an integrable Lie algebroid [8]. In fact, there exists a bundle of Lie groups (which may not be locally trivial nor Hausdorff) which integrates E.

2 Groupoid bundles and connections

2.1 Actions by Lie groupoids

In this section we begin by recalling several standard notions and terminology concerning groupoid bundles which play a central role in the proofs of some of our integrability results.

Let G be a fixed Lie groupoid, N a manifold and $\epsilon : N \to G_0$ a map. A left *action* of G on N *along* ϵ is given by a map $\vartheta : G_1 \times_{G_0} N \to N$ (we write $\vartheta(g, y) = gy$), defined on the pull-back $G_1 \times_{G_0} N = \{(g, y) \mid \alpha(g) = \epsilon(y)\}$, which satisfies the following identities: $\epsilon(gy) = \beta(g)$, $1_{\epsilon(y)}y = y$ and $g'(gy) = (g'g)y$, for any $g', g \in G_1$ and $y \in N$ with $\alpha(g') = \beta(g)$ and $\alpha(g) = \epsilon(y)$. For such an action one can form the *translation* groupoid $G \ltimes N$ over N, with $(G \ltimes N)_1 = G_1 \times_{G_0} N$, exactly as for groups (Example 1.2 (vi)). We define the quotient N/G as the space of orbits of the groupoid $G \ltimes N$. This space is in general not a manifold. A right action of G on N is defined analogously.

Suppose that we have a right G-action on a manifold P. If P is equipped with a map $\pi : P \to B$ and the action is fiberwise in the sense that $\pi(pg) = \pi(p)$ whenever pg is defined, then P is called a G-*bundle* over B. This G-bundle is said to be *principal* if π is a surjective submersion and the map $(\vartheta, pr_1) : P \times_{G_0} G_1 \to P \times_B P$ is a diffeomorphism. In this case the translation groupoid $P \rtimes G$ is isomorphic to the pair groupoid $P \times_B P$ over P. Note that $P/G \cong B$, so P/G is a manifold.

There is also a notion of a (left) action of a Lie groupoid G on another Lie groupoid H. It is given by two (left) actions of G on H_1 and on H_0, such that the groupoid structure maps of H are compatible with the actions by G [10]. If we denote the action maps on H_i by $\epsilon_i : H_i \to G_0$ and $\vartheta_i : G_1 \times_{G_0} H_i \to H_i$, $i = 0, 1$, this implies in particular that $\epsilon_0 \circ \alpha = \epsilon_1 = \epsilon_0 \circ \beta$. Thus the fibers $H_x = \epsilon_1^{-1}(x)$ are full subgroupoids of H over $\epsilon_0^{-1}(x)$, $x \in G_0$. These are Lie groupoids if ϵ_0 is a submersion, and for each arrow $g \in G_1(x', x)$ the action provides an isomorphism $H_{x'} \to H_x$ of Lie groupoids.

For such an action of G on H, one can form the *semi-direct product* groupoid

$$G \ltimes H$$

over H_0. For $y, y' \in H_0$, an arrow from y' to y in $G \ltimes H$ is a pair (g, h), where g is an arrow in $G(\epsilon_0(y'), \epsilon_0(y))$ and h is an arrow in $H(gy', y) \subset H_{\epsilon_0(y)}$. These arrows compose by the usual formula

$$(g, h)(g', h') = (gg', h(gh')) .$$

The groupoid $G \ltimes H$ has the natural structure of a Lie groupoid, as one sees, e.g. when the space of arrows is considered as the fibered product

$$H_1 \times_{G_0} G_1 = \{(h, g) \mid \epsilon_0(\beta(h)) = \beta(g)\} .$$

Lemma 2.1 *Consider an action of a Lie groupoid G on a Lie groupoid H. If H_0 is a principal G-bundle over B, then H_1/G is a Lie groupoid over $B \cong H_0/G$.*

PROOF. The only thing that has to be shown is that H_1/G is a manifold. We can specify the manifold structure locally in B, so it suffices to consider the case where $\pi : H_0 \to B$ has a section s. But then H_1/G is isomorphic to the pull-back of $\alpha : H_1 \to H_0$ along $s : B \to H_0$, hence is a manifold. Moreover, this manifold structure is independent of the choice of s, since by principality of the action on H_0 any two sections s and s' are related by a map $\theta : B \to G_1$ as $\theta(b)s(b) = s'(b)$ for all $b \in B$. Then the same multiplication by θ establishes a diffeomorphism between the pull-back of α along s and the one along s'. $\square$

REMARK. We denote the Lie groupoid H_1/G over B by H/G. The quotient morphism $H \to H/G$ induces for each $x \in H_0$ an isomorphism of α-fibers $\alpha^{-1}(x) \to \alpha^{-1}(\pi(x))$. More precisely, the square

$$
\begin{array}{ccc}
H_1 & \longrightarrow & H_1/G \\
\alpha \downarrow & & \downarrow \alpha \\
H_0 & \xrightarrow{\ \pi\ } & B = H_0/G
\end{array}
$$

is a pull-back of smooth manifolds.

2.2 Connections on principal groupoid bundles

Let G be a Lie groupoid with Lie algebroid $\mathfrak{g}$, and let $\pi : P \to B$ be a principal G-bundle along $\epsilon : P \to G_0$. For any $p \in P$, denote by $\mathcal{V}_p$ the space $\mathrm{Ker}((d\pi)_p)$ of vertical tangent vectors at $p \in P$. Thus $\mathcal{V}$ is an integrable subbundle of $T(P)$. The diffeomorphism $L_p : G(\epsilon(p), \text{-}) \to P_{\pi(p)}$, given by $L_p(g) = pg^{-1}$, induces and isomorphism $dL_p : \mathfrak{g}_{\epsilon(p)} \to \mathcal{V}_p$.

Recall that a *local bisection* of G is a local section $s : U \to G_1$ of the source map α, defined on an open subset U of G_0, such that $\beta \circ s$ is an open embedding. We say that s is a local bisection of G through $g \in G_1$ if $g \in s(U)$. It is easy to see that there exist local bisections through any arrow of G. Such a bisection s induces a diffeomorphism $R_s : \epsilon^{-1}(\beta(s(U)) \to \epsilon^{-1}(U)$ by $R_s(p) = ps(x)$, where $x \in G_0$ is the unique point with $\beta(s(x)) = \epsilon(p)$. If ξ is in the kernel of $(d\epsilon)_p$, then $(dR_s)_p(\xi)$ depends only on the value g of s with $\beta(g) = \epsilon(p)$, and we will write $\xi g = (dR_s)_p(\xi)$. Indeed, in this case one may define ξg as the image of $(\xi, 0)$ along the derivative of the action $P \times_{G_0} G_1 \to P$ at (p, g).

Let $\mathcal{F}$ be a foliation of B. Then $\pi^*(\mathcal{F})$ is a foliation of P. An $\mathcal{F}$-*partial connection* on P is a subbundle $\mathcal{H}$ of $\pi^*(\mathcal{F}) \subset T(P)$ which satisfies the following conditions:

(i) $\pi^*(\mathcal{F}) = \mathcal{V} \oplus \mathcal{H}$,

(ii) $(d\epsilon)(\mathcal{H}) = 0$, and

(iii) $\mathcal{H}_{pg} = \mathcal{H}_p g$ for any $p \in P$ and $g \in G$ with $\epsilon(p) = \beta(g)$.

Note that $\mathcal{H}_p g$ is well-defined precisely because of the condition (ii) above. The connection $\mathcal{H}$ is called *flat* if it is integrable. With a given connection $\mathcal{H}$, any

tangent vector $\xi \in \pi^*(\mathcal{F})_p$ has a unique decomposition as a sum $\xi = \xi^v + \xi^h$ of its vertical and horizontal parts.

There is the associated *partial connection form* ω on P with values in $\mathfrak{g}$, given by

$$\omega_p(\xi) = dL_p^{-1}(\xi^v) \in \mathfrak{g}_{\epsilon(p)}$$

for any $\xi \in \pi^*(\mathcal{F})_p$.

Proposition 2.2 *The partial connection form ω associated to a connection $\mathcal{H}$ on a principal G-bundle P has the following properties:*

1. $\omega : \pi^*(\mathcal{F}) \to \mathfrak{g}$ *is a map of vector bundles over $\epsilon : P \to G_0$,*

2. $\omega_p \circ dL_p = id$,

3. $\mathrm{Ker}(\omega_p) \subset \mathrm{Ker}((d\epsilon)_p)$, *and*

4. $R_s^*\omega = \mathrm{Ad}(inv \circ s) \circ \omega$, *for any local bisection $s : U \to G_1$ of G.*

REMARK. Explicitly, the condition (4) means that for any $x \in U$, any $p \in \epsilon^{-1}(\beta(s(x)))$ and any $\xi \in \pi^*(\mathcal{F})_p$

$$(R_s^*\omega)_p(\xi) = \omega_{ps(x)}(dR_s(\xi)) = (dL_s)_{1_x}^{-1}\omega_p(\xi)s(x) = \mathrm{Ad}(inv \circ s)(\omega_p(\xi)) \ .$$

Here dL_s is the derivative of the diffeomorphism $L_s : \beta^{-1}(U) \to \beta^{-1}(\beta(s(U)))$ given by $L_s(g) = s(\beta(g))g$. The map ω_p is completely determined by its restriction to the subspace of those ξ which are in the kernel of $d\epsilon$, in which case $\rho(\omega_p(\xi)) = 0$, and the condition (4) may be expressed simply as $(R_{s(x)}^*\omega)(\xi) = \mathrm{Ad}(s(x)^{-1})(\omega(\xi))$. Conversely, any ω satisfying the conditions above determines a connection by

$$\mathcal{H}_p = \mathrm{Ker}(\omega_p) \ .$$

Proposition 2.3 *Let P be a principal G-bundle over B, $\mathcal{F}$ a foliation of B and $\mathcal{H}$ a flat $\mathcal{F}$-partial connection on P. Then each leaf of $\mathcal{H}$ projects by a covering projection to a leaf of $\mathcal{F}$.*

PROOF. Each leaf $\tilde{L}$ of $\mathcal{H}$ clearly projects by a local diffeomorphism to a leaf L of $\mathcal{F}$. Moreover, $\tilde{L}$ lies in a fiber $\epsilon^{-1}(x)$. Let $\mathrm{Iso}(\tilde{L})$ be the isotropy group of $\tilde{L}$, i.e.

$$\mathrm{Iso}(\tilde{L}) = \{g \in G(x, x) \,|\, \tilde{L}g = \tilde{L}\} \ .$$

The group $\mathrm{Iso}(\tilde{L})$, equipped with the discrete topology, acts freely and properly discontinuously on $\tilde{L}$, which shows that π in fact restricts to a covering projection $\tilde{L} \to L$. $\square$

3 Lie theory for groupoids

3.1 The source-simply connected cover

Every finite dimensional Lie algebra is the Lie algebra of a unique connected simply connected Lie group. Something similar holds for *integrable* Lie algebroids.

Definition 3.1 *A Lie groupoid G is said to be source-connected if $\alpha^{-1}(x)$ is connected for any $x \in G_0$. It is said to be source-simply connected if each $\alpha^{-1}(x)$ is connected and simply connected.*

Example 3.2 The monodromy groupoid $\mathrm{Mon}(M, \mathcal{F})$ of a foliated manifold is source-simply connected.

The following proposition is proved in [13] for the special case of transitive Lie groupoids.

Proposition 3.3 *Let G be a Lie groupoid. There exists a source-simply connected Lie groupoid $\tilde{G}$ over G_0 and a morphism of Lie groupoids $\pi : \tilde{G} \to G$ over G_0, inducing an isomorphism $\mathcal{A}(\tilde{G}) \to \mathcal{A}(G)$.*

REMARK. The covering groupoid $\tilde{G}$ is essentially unique, see Proposition 3.5 below.

PROOF. For each $x \in G_0$ let $\alpha^{-1}(x)^{(0)}$ be the connected component of $\alpha^{-1}(x)$ which contains 1_x. It is well-known and easy to see that the union of these $\alpha^{-1}(x)^{(0)}$ form a source-connected open subgroupoid $G^{(0)}$ of G over G_0 having the same Lie algebroid. Thus, to prove the proposition we can first replace G by $G^{(0)}$, and hence assume that G is source-connected.

Now let $\mathcal{F}$ be the foliation of G_1 given by the fibers of α, and let $\mathrm{Mon}(G_1, \mathcal{F})$ be its monodromy groupoid over G_1. The space G_1 is a principal (right) G-bundle over G_0 (with structure maps β for π and α for ϵ) and this principal action maps leaves to leaves. Thus G also acts on the monodromy groupoid. By Lemma 2.1 we can form the quotient Lie groupoid $\tilde{G} = \mathrm{Mon}(G_1, \mathcal{F})/G$, which is a groupoid over G_0. Since any monodromy groupoid is source-simply connected and $\mathrm{Mon}(G_1, \mathcal{F})$ has the same α-fibers as its quotient by G by the remark after Lemma 2.1, $\tilde{G}$ is again source-simply connected. Finally, the morphism of Lie groupoids

$$F : \mathrm{Mon}(G_1, \mathcal{F}) \longrightarrow G$$

given by $\beta : G_1 \to G_0$ on objects and by $F(\sigma) = \sigma(1)\sigma(0)^{-1}$ on arrows of $\mathrm{Mon}(G_1, \mathcal{F})$ (where σ is the homotopy class of a path inside a leaf of $\mathcal{F}$) factors to give the required map $\tilde{G} \to G$. $\square$

3.2 Integrability of subalgebroids

Let $\mathfrak{g}$ be a Lie algebroid over M, and let N be an immersed submanifold of M. A *subalgebroid* of $\mathfrak{g}$ over N is a subbundle $\mathfrak{h}$ of the restriction $\mathfrak{g}|_N$ with a Lie algebroid structure such that the inclusion $\mathfrak{h} \to \mathfrak{g}$ is a morphism of Lie algebroids.

The same methods involved in the construction of the source-simply connected groupoid $\tilde{G}$ can be used to prove the following integrability result. (We point out that the case of transitive groupoids was proved earlier in [13], and a local integrability result involving micro-differentiable groupoids was proved by Almeida, see [13, p. 158].)

Proposition 3.4 *Any subalgebroid of an integrable algebroid is integrable.*

PROOF. Consider a Lie groupoid G with associated Lie algebroid $\mathfrak{g}$ and a subalgebroid $\mathfrak{h}$ over H_0 of $\mathfrak{g}$. Denote by $I : \mathfrak{h} \to \mathfrak{g}$ the inclusion of Lie algebroids over the injective immersion $\iota : H_0 \to G_0$, and write

$$M = H_0 \times_{G_0} G_1$$

for the pull-back of $\beta : G_1 \to G_0$ along ι. Consider the foliation $\mathcal{F}$ of M given by

$$\mathcal{F}_{(y,g)} = \{(\rho(u), I(u)g) \mid u \in \mathfrak{h}_y\}$$

The composition of the groupoid G defines on the manifold M the structure of a principal G-bundle over H_0, and the foliation $\mathcal{F}$ is preserved by the G-action. Thus the monodromy groupoid $\mathrm{Mon}(M, \mathcal{F})$ also carries a right G-action, and by Lemma 2.1 we obtain a quotient groupoid

$$H = \mathrm{Mon}(M, \mathcal{F})/G$$

over H_0. Its Lie algebroid is easily seen to be isomorphic to $\mathfrak{h}$. $\square$

REMARK. By the integrability of morphisms between integrable Lie algebroids (see Proposition 3.5 below), there is a map $H \to G$ which is in fact an immersion. In this sense the subalgebroid is integrated by an immersed groupoid.

3.3   Integrability of morphisms between Lie algebroids

As an application of Proposition 2.3, we shall give a quick proof of the fact that any morphism of integrable Lie algebroids can be integrated to a unique morphisms of the integral Lie groupoids, provided that the domain groupoid is source-simply connected. This fact has been proved earlier by Mackenzie and Xu [17].

Proposition 3.5 *Let G and H be Lie groupoids, with H source-simply connected, and let $\Phi : \mathfrak{h} \to \mathfrak{g}$ be a morphism of their Lie algebroids over $\phi : H_0 \to G_0$. Then there exists a unique morphism of Lie groupoids $F : H \to G$ with $F_0 = \phi$ which integrates Φ, i.e. $\mathcal{A}(F) = \Phi$.*

PROOF. Let $P = H_1 \times_{G_0} G_1$ be the pull-back of $\beta : G_1 \to G_0$ along the map $\phi \circ \beta : H_1 \to G_0$. Thus P is a (trivial) principal G-bundle over H_1, with the obvious right action with respect to the map $\epsilon = \alpha \circ pr_2$. Let $\mathcal{F}$ be the foliation of H_1 by the α-fibers. Now define a partial connection $\mathcal{H}$ on P by

$$\mathcal{H}_{(h,g)} = \{(vh, \Phi(v)g) \mid v \in \mathfrak{h}_{\beta(h)}\} \, .$$

This is indeed a flat connection on E because Φ preserves the bracket. Now take any $y \in H_0$, and denote by $\tilde{L}_y$ the leaf of $\mathcal{H}$ through the point $(1_y, 1_{\phi(y)})$. By Proposition 2.3, $\tilde{L}_y$ is a covering space over the corresponding leaf of $\mathcal{F}$, i.e. the α-fiber $\alpha^{-1}(y)$. Since the α-fibers of H are simply connected, the projection $\tilde{L}_y \to \alpha^{-1}(y)$ is in fact a diffeomorphism. Denote by ν_y the inverse of this diffeomorphism. Now the union of the maps ν_y gives us a map $\nu : H_1 \to P$. Observe that this map is smooth, since it can be described as an extension of the smooth transversal section $H_0 \to P$ by holonomy. Then we take F_1 to be the composition $F_1 = pr_2 \circ \nu : H \longrightarrow G$. In particular, F_1 maps $\alpha^{-1}(y)$ to $\alpha^{-1}(\phi(y))$. It is easy to see that F_1 together with $F_0 = \phi$ gives a morphism of Lie groupoids $F : H \to G$. Now for any $v \in \mathfrak{h}_y$ we have $d(F_1)(v) = d(pr_2)(v, \Phi(v)) = \Phi(v)$, hence $\mathcal{A}(F) = \Phi$. $\square$

3.4 Integrability of actions by Lie groupoids on Lie algebroids

Consider a Lie algebroid $\mathfrak{h}$ over N. If $q : N \to B$ is a submersion which annihilates the anchor of H (i.e. $dq \circ \rho = 0$), then it follows from the Leibniz identity (Section 1.1) that for two sections $Y, Y' \in \Gamma\mathfrak{h}$, the value $[Y, Y']_y$ of the bracket in a point $y \in N$ only depends on the restrictions of Y and Y' to the fiber $q^{-1}(q(y))$. Thus for $b \in B$ the fiber $\mathfrak{h}_b = \mathfrak{h}|_{q^{-1}(b)}$ is a Lie subalgebroid of $\mathfrak{h}$, and we can think of $\mathfrak{h}$ as a *family* of Lie algebroids over B. For such a family $\mathfrak{h} \to N \to B$, the pull-back along any map $\phi : B' \to B$ is a Lie algebroid over $B' \times_B H_0$, which we denote by $\phi^*(\mathfrak{h})$ – it is a family of Lie algebroids over B'.

Now suppose that G is a Lie groupoid, and $\mathfrak{h}$ a family of Lie algebroids over G_0, with respect to a surjective submersion $q : H_0 \to G_0$ annihilating the anchor of $\mathfrak{h}$, as before. A left *action* of G on $\mathfrak{h}$ *along* q is a morphism of Lie algebroids

$$\vartheta : \alpha^*(\mathfrak{h}) \longrightarrow \beta^*(\mathfrak{h})$$

over G_1, satisfying the unit and cocycle conditions expressing the unit and associativity laws of an action. Thus, for an arrow $g \in G(x', x)$, the fiber of ϑ over g is a Lie algebroid morphism

$$\vartheta_g : \mathfrak{h}_{x'} \longrightarrow \mathfrak{h}_x \qquad \vartheta_g(u) = gu$$

and the unit and cocycle conditions state that $\vartheta_{1_x} = id$ and $\vartheta_g \vartheta_{g'} = \vartheta_{gg'}$. In particular, the effect of the map ϑ on the base manifolds of the algebroids is a map $G_1 \times_{G_0} H_0 \to H_0 \times_{G_0} G_1$ over G_1, and its composition with pr_1 defines a G-action on the space H_0. For $g \in G(x', x)$ and $y \in q^{-1}(x')$ this action is denoted by $gy \in q^{-1}(x)$ as before. Thus if $u \in \mathfrak{h}_y$ is a point in the algebroid $\mathfrak{h}$ over y, then $gu \in \mathfrak{h}_{gy}$.

For example, a left action by a Lie groupoid G on a Lie groupoid H induces an action of G on the Lie algebroid $\mathfrak{h}$ of H, by taking the derivative of the action on arrows.

We will now show that if $\mathfrak{h}$ is integrable, then so is any action by G on $\mathfrak{h}$, i.e. any such action comes from an action by G on the source-simply connected integral H of $\mathfrak{h}$.

Theorem 3.6 *Let $\mathfrak{h}$ be the Lie algebroid of an source-simply connected Lie groupoid H. Any action of a Lie groupoid G on the Lie algebroid $\mathfrak{h}$ can be integrated uniquely to an action of G on H.*

PROOF. As before, we denote by $q : H_0 \to G_0$ the structure map of this action. Note that since $dq \circ \rho = 0$, the map $q \circ \beta$ is locally constant on the α-fibers of H, and hence it is constant since H is source-connected. Thus $q \circ \alpha = q \circ \beta$. Now consider the pull-back manifold

$$M = G_1 \times_{G_0} H_1 = \{(g, h) \,|\, \alpha(g) = q(\alpha(h))\} \,.$$

This manifold is equipped with a foliation $\mathcal{F}$, whose leaves are the fibers of the map $id \times \alpha : M = G_1 \times_{G_0} H_1 \to G_1 \times_{G_0} H_0$. Over M there is a principal (right) H-bundle

$$P = G_1 \times_{G_0} H_1 \times_{H_0} H_1 = \{(g, h, h') \,|\, (g, h) \in M, \, g\beta(h) = \beta(h')\}$$

with structure maps $\alpha \circ pr_3 : P \to H_0$ and action $(g, h, h')h'' = (g, h, h'h'')$. The foliation $\mathcal{F}$ of M lifts along the projection $\pi : P \cong M \times_{H_0} H_1 \to M$ to a foliation $\mathcal{H}$ of P, whose tangent space $\mathcal{H}_{(g,h,h')} \subset T_{(g,h,h')}(P)$ is defined in terms of the action of G on $\mathfrak{h}$ by

$$\mathcal{H}_{(g,h,h')} = \{(0g, uh, (gu)h') \mid u \in \mathfrak{h}_{\beta(h)}\} .$$

To explain this notation, let $g \in G(x', x)$ and $y = \beta(h)$, $gy = \beta(h')$. Then $0g$ is the zero tangent vector in $T_g(G_1)$, and $uh \in T_h^\alpha(H_1)$. Furthermore, $gu \in \mathfrak{h}_{gy}$ and hence $(gu)h' \in T_{h'}^\alpha(H_1)$.

Note that, since the action of G on $\mathfrak{h}$ preserves the bracket, this subbundle $\mathcal{H}$ of $T(P)$ is integrable, so that $\mathcal{H}$ is indeed a foliation of P. It is clear from this definition that $\mathcal{H}$ is a partial flat connection on the principal H-bundle P over the foliation $(M, \mathcal{F})$. So by Proposition 2.3 and the fact that the leaves of $\mathcal{F}$ are simply connected, the projection $\pi : P \to M$ of the principal bundle restricts to a diffeomorphism $L \to \pi(L)$ from any leaf L of $\mathcal{H}$ to the leaf $\pi(L)$ of $\mathcal{F}$. Now consider the complete transversal sections S of $(M, \mathcal{F})$ and T of $(P, \mathcal{H})$, defined by

$$S = \{(g, 1_y) \mid g \in G_1, \, y \in H_0, \, \alpha(g) = q(y)\}$$

and

$$T = \{(g, 1_y, 1_{y'}) \mid g \in G_1, \, y, y' \in H_0, \, \alpha(g) = q(y), \, gy = y'\} .$$

Let $\phi : M \to P$ be the unique section of π which sends T into S by $\phi(g, 1_y) = (g, 1_y, 1_{q(y)})$ and which maps the leaf through $(g, 1_y)$ to the leaf through $\phi(g, 1_y)$. Then

$$\vartheta = pr_3 \circ \phi : G_1 \times_{G_0} H_1 \longrightarrow H_1$$

is the action map which integrates the given action of G on $\mathfrak{h}$. Indeed, to see that it respects the composition in G and in H, note that for a fixed $g \in G(x', x)$ the map $\vartheta_g = \vartheta(g, \text{-})$ is the unique Lie groupoid morphism $H_{q^{-1}(x')} \to H_{q^{-1}(x)}$, integrating the algebroid map $\mathfrak{h}_{x'} \to \mathfrak{h}_x$ which sends u to gu (see Proposition 3.5). $\square$

4 Derivations and infinitesimal actions

4.1 Derivations on Lie algebroids

Let M be a manifold , and $\mathfrak{X}(M)$ the Lie algebra of vector fields on M. Recall that a derivation on $\mathfrak{X}(M)$ is an $\mathbb{R}$-linear map $D : \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfying

$$D([X, X']) = [D(X), X'] + [X, D(X')]$$

for any two vector fields $X, X' \in \mathfrak{X}(M)$. Each vector field V on M defines an (inner) derivation D on $\mathfrak{X}(M)$ given by $D(X) = [V, X]$, and in fact any derivation on $\mathfrak{X}(M)$ is of this form for a unique vector field $V \in \mathfrak{X}(M)$ [27]. In this section we will prove a similar result for derivations on the Lie algebra of sections of a Lie algebroid.

Definition 4.1 *A derivation on a Lie algebroid $\mathfrak{g}$ over M is a pair (D, V) consisting of an $\mathbb{R}$-linear map $D : \Gamma\mathfrak{g} \to \Gamma\mathfrak{g}$ and a vector field V on M such that*

(i) $D([X, X']) = [D(X), X'] + [X, D(X')]$,

(ii) $D(fX) = fD(X) + V(f)X$ *and*

(iii) $\rho(D(X)) = [V, \rho(X)]$

for any $X, X' \in \Gamma\mathfrak{g}$ *and* $f \in C^\infty(M)$.

REMARK. If the rank of $\mathfrak{g}$ is not zero, the properties (i) and (ii) imply property (iii). The vector space of all derivations on $\mathfrak{g}$, denoted by $\mathrm{Der}(\mathfrak{g})$, is a Lie algebra with respect to the bracket $[(D, V), (D', V')] = (D \circ D' - D' \circ D, [V, V'])$.

Examples 4.2 (i) For a Lie algebra (viewed as a Lie algebroid over a one point space) we recover the usual notion of a derivation.

(ii) Suppose that $E \to M$ is a vector bundle, viewed as a Lie algebroid as in Example 1.1 (iii). A derivation on this Lie algebroid consists of a vector field V on M and a partial connection (a covariant differential operator) $D = \nabla_V$ on E.

(iii) Let $\mathcal{F}$ be a foliation of a manifold M, viewed as a Lie algebroid over M with injective anchor map. The Lie algebra of derivations $\mathrm{Der}(\mathcal{F})$ can be identified with the Lie algebra $L(M, \mathcal{F})$ of projectable vector fields on $(M, \mathcal{F})$ (see [18]).

For a Lie groupoid G, there is an associated tangent Lie groupoid $T(G)$ over $T(G_0)$. Its manifold of arrows is $T(G_1)$, while the source and the target maps $T(G_1) \to T(G_0)$ and the multiplication map $T(G_1) \times_{T(G_0)} T(G_1) \cong T(G_1 \times_{G_0} G_1) \to T(G_1)$ are the derivatives of those of G. The bundle projections $\pi_i : T(G_i) \to G_i$, $i = 0, 1$, define a morphism of Lie groupoids

$$T(G) \longrightarrow G .$$

Definition 4.3 *A multiplicative vector field on a Lie groupoid G is a morphism of Lie groupoids $W : G \to T(G)$, which is a section of the projection $T(G) \to G$.*

REMARK. Multiplicative vector fields were studied in [16]. A multiplicative vector field W on G is, in other words, a pair of vector fields W_0 on G_0 and W_1 on G_1, such that W_1 is projectable to W_0 along both α and β, W_0 is projectable to W_1 along $uni : G_0 \to G_1$, and $(W_1, W_1) \in \mathfrak{X}(G_1 \times_{G_0} G_1)$ is projectable to W_1 along the multiplication of G. The last two conditions mean that

$$(W_1)_{1_x} = 1_{(W_0)_x} = d(uni)((W_0)_x) \qquad \text{and} \qquad (W_1)_{gg'} = (W_1)_g(W_1)_{g'} ,$$

the latter composition being the one in $T(G)$. Note that W_0 is determined by W_1, in fact the restriction of W_1 to G_0 is tangent to G_0 and can be identified with W_0. Therefore we will simply write W for W_1 and identify W_0 with $W_1|_{G_0}$.

The Lie brackets of vector fields on G_0 and G_1 respect projectability along the maps between G_0 and G_1 and hence define a Lie bracket on multiplicative vector fields on G. In this way, the multiplicative vector fields on G form a Lie algebra, denoted by $\mathfrak{X}^\mu(G)$.

Lemma 4.4 *For any Lie groupoid G we have $[\mathfrak{X}^\mu(G), \mathfrak{X}^\alpha_{inv}(G)] \subset \mathfrak{X}^\alpha_{inv}(G)$.*

PROOF. Let $W \in \mathfrak{X}^\mu(G)$ and $X \in \mathfrak{X}^\alpha_{inv}(G)$. The invariance of X is equivalent to the condition that the vector field $(X, 0)$ on $G_1 \times_{G_0} G_1$ is projectable to X along μ. Since W is projectable along α and X is tangent to the fibers of α, $[W, X]$ is tangent to the fibers of α as well. Since both (W, W) and $(X, 0)$ are projectable along μ, so is $[(W, W), (X, 0)] = ([W, X], 0)$, and $d\mu \circ ([W, X], 0) = [W, X] \circ \mu$.
□

Let G be a Lie groupoid and $\mathfrak{g}$ the Lie algebroid associated to G. Any multiplicative vector field W on G gives us a derivation $\mathcal{L}(W) = (\mathcal{L}_W, W|_{G_0})$ on $\mathfrak{g}$ by

$$\mathcal{L}_W(X) = [W, X] \qquad X \in \Gamma\mathfrak{g} \cong \mathfrak{X}^\alpha_{inv}(G) .$$

Indeed, Lemma 4.3 implies that the image of $\mathcal{L}_W$ is in $\Gamma\mathfrak{g}$, while it is easy to check that the properties (i), (ii) and (iii) in Definition 4.1 are satisfied. Moreover, $\mathcal{L}$ is a morphism of Lie algebras

$$\mathcal{L} : \mathfrak{X}^\mu(G) \longrightarrow \mathrm{Der}(\mathfrak{g}) .$$

The following theorem follows from the results of [16]:

Theorem 4.5 *If G is a source-simply connected Lie groupoid, then the map $\mathcal{L}$ is an isomorphism of Lie algebras.*

PROOF. First we will show that $\mathcal{L}$ is injective. Let $W \in \mathrm{Ker}\mathcal{L}$. In particular, $d\alpha \circ W = d\beta \circ W = 0$, $W|_{G_0} = 0$ and $[W, \mathfrak{X}^\alpha_{inv}(G)] = 0$. Take any $g \in G_1$, and consider the fiber $G(\alpha(g), \text{-})$. Since any tangent vector on $G(\alpha(g), \text{-})$ can be extended to an α-vertical invariant vector field, the condition $[W, \mathfrak{X}^\alpha_{inv}(G)] = 0$ implies that the subset of zeros of W is open in $G(\alpha(g), \text{-})$. But $G(\alpha(g), \text{-})$ is connected and $W_{\alpha(g)} = 0$, thus $W_g = 0$.

Next we will prove that $\mathcal{L}$ is surjective. Let (D, V) be a derivation on the Lie algebroid $\mathfrak{g}$ associated to G. Take any $u \in \mathfrak{g}_x$. For any $v \in \mathfrak{g}_x$ let $\tau(u, v) \in \mathrm{Ker}((d\pi)_u) \subset T_u(\mathfrak{g})$ be given by

$$\tau(u, v)(f) = \left.\frac{d}{dt}\right|_{t=0} f(u + tv) \qquad f \in C^\infty(\mathfrak{g}) .$$

In other words, the map $\tau(u, \text{-})$ is the natural isomorphism between $\mathfrak{g}_x$ and $\mathrm{Ker}((d\pi)_u)$. Now define

$$\Xi(u) = (dX)_x(V_x) - \tau(u, D(X)_x) \in T_u(\mathfrak{g}) ,$$

for any section $X \in \Gamma\mathfrak{g}$ satisfying $X_x = u$. The property (ii) of D (Definition 4.1) implies that the definition of $\Xi(u)$ does not depend on the choice of X, so we get a map

$$\Xi : \mathfrak{g} \longrightarrow T(\mathfrak{g}) .$$

Moreover, Ξ is a bundle map over V by [16, Proposition 2.5]. Now recall from [15] that $T(\mathfrak{g})$ has a natural structure of a Lie algebroid over $T(G_0)$ and that there is a natural isomorphism of Lie algebroids $j : T(\mathfrak{g}) \to \mathcal{A}(T(G))$ over $T(G_0)$. It follows from [16, Theorem 4.4] that Ξ is a morphism of Lie algebroids, hence $\tilde{\Xi} = j \circ \Xi : \mathfrak{g} \to \mathcal{A}(T(G))$ is a morphism of Lie algebroids over V as well. Since G is source-simply connected, Proposition 3.5 implies that $\tilde{\Xi}$ can be integrated to a unique morphism of Lie groupoids

$$W : G \longrightarrow T(G) .$$

Note that W is a multiplicative vector field on G because $\tilde{\tilde{\Xi}}$ is a section of the projection $\mathcal{A}(T(G)) \to \mathfrak{g}$, and that $W|_{G_0} = V$. Finally, [16, Theorem 3.9] implies that $\mathcal{L}_W = D$. $\square$

4.2 Infinitesimal actions

Let $\mathfrak{h}$ be a Lie algebroid over N and $q : N \to M$ a surjective submersion. Thus the Lie algebra $\Gamma\mathfrak{h}$ has the structure of a $C^\infty(M)$-module induced by the composition with q. Assume that $dq \circ \rho = 0$, i.e. that $\mathfrak{h}$ is a family of Lie algebroids over M. This implies that $\rho(Y)(f \circ q) = 0$ for any $Y \in \Gamma\mathfrak{h}$ and any $f \in C^\infty(M)$, and hence the Lie bracket on $\Gamma\mathfrak{h}$ is $C^\infty(M)$-bilinear. Furthermore, the Lie algebra of derivations $\mathrm{Der}(\mathfrak{h})$ on $\mathfrak{h}$ becomes a $C^\infty(M)$-module via $f(D, V) = ((f \circ q)D, (f \circ q)V)$. If H is a source-simply connected integral of $\mathfrak{h}$, the Lie algebra $\mathfrak{X}^\mu(H)$ of multiplicative vector fields on H is also a $C^\infty(M)$-module and the isomorphism $\mathcal{L}$ of Theorem 4.5 is $C^\infty(M)$-linear.

Definition 4.6 *Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebroids over M respectively N, and let $q : N \to M$ be a surjective submersion such that $dq \circ \rho = 0$. An (infinitesimal) action of $\mathfrak{g}$ on $\mathfrak{h}$ along q is a homomorphism of Lie algebras $\nabla : \Gamma\mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$, $\nabla(X) = (\nabla_X, R(X))$, which is $C^\infty(M)$-linear and for which each $R(X)$ is projectable to $\rho(X)$ along q.*

REMARK. In particular, $R : \Gamma\mathfrak{g} \to \mathfrak{X}(N)$ is a homomorphism of Lie algebras satisfying $\nabla_X(f'Y) = f'\nabla_X Y + R(X)(f')Y$ for any $X \in \Gamma\mathfrak{g}$, $Y \in \Gamma\mathfrak{h}$ and $f' \in C^\infty(N)$. Furthermore, the $C^\infty(M)$-linearity of ∇ implies that $\nabla_{fX}(Y) = (f \circ q)\nabla_X(Y)$ and $R(fX) = (f \circ q)R(X)$. The second equality in fact follows from the first if the rank of $\mathfrak{h}$ is not zero. The projectability of $R(X)$ to $\rho(X)$ along q means that $R(X)(f \circ q) = \rho(X)(f) \circ q$ for any $f \in C^\infty(M)$ and $X \in \Gamma\mathfrak{g}$. We shall say that ∇ is an action *over R*. Our definition is clearly equivalent with [10, Definition 3.6].

Examples 4.7 (i) If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, we recover the usual notion of an action [26].

(ii) If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{h}$ is the tangent bundle of a manifold N, an infinitesimal action of $\mathfrak{g}$ on $\mathfrak{h}$ is the same thing as an infinitesimal action of $\mathfrak{g}$ on N (Example 1.1 (vi)). More generally, if $\mathfrak{h}$ is a foliation $\mathcal{F}$ of a manifold N (Example 1.1 (v)), an infinitesimal action of $\mathfrak{g}$ on $\mathcal{F}$ is a Lie algebra map $\mathfrak{g} \to L(N, \mathcal{F})$ into the projectable vector fields on $(N, \mathcal{F})$.

(iii) Let $E \to M$ be a vector bundle over a foliated manifold $(M, \mathcal{F})$. We can consider $\mathcal{F}$ and E as Lie algebroids over the same manifold M (Examples 1.1 (iv) and (v)), and an action of $\mathcal{F}$ on E along the identity map is the same thing as an affine flat $\mathcal{F}$-partial connection on E.

Let ∇ be an action of $\mathfrak{g}$ on $\mathfrak{h}$ along $q : N \to M$ over R. Then the *semi-direct product* [10] of $\mathfrak{g}$ and $\mathfrak{h}$ with respect to ∇ is a Lie algebroid $\mathfrak{g} \ltimes \mathfrak{h}$ over N given as follows: as a vector bundle it is the direct sum $q^*\mathfrak{g} \oplus \mathfrak{h}$, the anchor is given by

$$\rho(q^*X \oplus Y) = R(X) + \rho(Y) ,$$

and the bracket by

$$[q^*X \oplus Y, q^*X' \oplus Y'] = q^*[X, X'] \oplus ([Y, Y'] + \nabla_X(Y') - \nabla_{X'}(Y)) .$$

Here $q^*X = (id, X \circ q) \in \Gamma q^*\mathfrak{g}$. Since the sections of this form span $\Gamma q^*\mathfrak{g}$ as a $C^\infty(N)$-module, we can extend the definition of the anchor and of the bracket to all the sections of $q^*\mathfrak{g} \oplus \mathfrak{h}$ by the $C^\infty(N)$-linearity of the anchor and by the Leibniz identity.

Observe that we have an exact sequence

$$0 \longrightarrow \mathfrak{h} \stackrel{j}{\longrightarrow} \mathfrak{g} \ltimes \mathfrak{h} \stackrel{\pi}{\longrightarrow} q^*\mathfrak{g} \longrightarrow 0$$

of vector bundles over N. The action of $\mathfrak{g}$ on $\mathfrak{h}$ can be recovered from this sequence by

$$j(\nabla_X(Y)) = [(q^*X, 0), (0, Y)] .$$

Semi-direct products are related to split exact sequences in the usual way. Explicitly, suppose that $\mathfrak{k}$ is a Lie algebroid over N which fits into an exact sequence

$$0 \longrightarrow \mathfrak{h} \stackrel{j}{\longrightarrow} \mathfrak{k} \stackrel{\pi}{\longrightarrow} q^*\mathfrak{g} \longrightarrow 0$$

of vector bundles over N. Suppose that j is a map of Lie algebroids over N, and that π is given by map of Lie algebroids $\mathfrak{k} \to \mathfrak{g}$ over q. Consider a splitting of this exact sequence by a map $i : q^*\mathfrak{g} \to \mathfrak{k}$ of vector bundles. The map i defines a "connection" $\nabla : \Gamma\mathfrak{g} \otimes \Gamma\mathfrak{h} \to \Gamma\mathfrak{h}$, $\nabla(X \otimes Y) = \nabla_X(Y)$, by

$$j(\nabla_X(Y)) = [i(q^*X), j(Y)]_{(\mathfrak{k})} .$$

The curvature 2-form of this connection is the map $\kappa : \Gamma\mathfrak{g} \wedge \Gamma\mathfrak{g} \to \Gamma\mathfrak{h}$ given by

$$j(\kappa(X, X')) = [i(q^*X), i(q^*X')]_{(\mathfrak{k})} - i(q^*[X, X']_{(\mathfrak{g})}) .$$

The connection ∇ is *flat* if $[\kappa(X, X'), Y] = 0$ for every $Y \in \Gamma\mathfrak{h}$. In particular, this is the case if $\kappa = 0$, i.e. if i preserves the bracket. In this case, ∇ is an action of $\mathfrak{g}$ on $\mathfrak{h}$ along q as defined above, with $R(X) = \rho(i(q^*X))$, and $\mathfrak{k}$ is isomorphic to $\mathfrak{g} \ltimes \mathfrak{h}$.

Examples 4.8 (i) Let P be a principal H-bundle over M for a Lie group H, and let G be the gauge groupoid over M (Example 1.2 (vii)), with Lie algebroid $\mathfrak{g}$. There is an exact "Atiyah" sequence over M,

$$0 \longrightarrow \mathfrak{h}^{tw} \longrightarrow \mathfrak{g} \longrightarrow T(M) \longrightarrow 0 ,$$

where $\mathfrak{h}^{tw}$ is the bundle of Lie algebras over M obtained by twisting the trivial bundle $M \times \mathfrak{h}$ by the adjoint action of the cocycle defining the principal bundle [2]. Here $\mathfrak{h}$ is the Lie algebra associated to H. An Ehresmann connection is the same thing as a splitting of this exact sequence. Flat connections correspond to actions of $T(M)$ on $\mathfrak{h}^{tw}$, and represent $\mathfrak{g}$ as a semi-direct product $T(M) \ltimes \mathfrak{h}^{tw}$.

(ii) If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebroids with injective anchor maps, then the semi-direct product $\mathfrak{g} \ltimes \mathfrak{h}$ of an infinitesimal action of $\mathfrak{g}$ on $\mathfrak{h}$ along $q : N \to M$ again has injective anchor. (Indeed, if $X \in \Gamma\mathfrak{g}$ and $Y \in \Gamma\mathfrak{h}$ are such that $(R(X) + \rho(Y))_y = 0$ for a point $y \in N$, then $0 = dq(R(X) + \rho(Y))_y = \rho(X)_{q(y)}$, whence $X_{q(y)} = 0$. Then also $R(X)_y = 0$ because R is $C^\infty(M)$-linear, whence $\rho(Y)_y = 0$ so $Y_y = 0$.) Thus, a semi-direct product of two foliations is again a foliation.

5 Integrability of semi-direct products

5.1 Infinitesimal actions on foliations

In this section we consider infinitesimal actions of a Lie algebroid $\mathfrak{g}$ on a Lie algebroid $\mathfrak{h}$ in the special case where $\mathfrak{h}$ is a foliation $\mathcal{F}$ of a manifold N (Example 1.1 (v)). As noted before, such an algebroid $\mathcal{F}$ is always integrable, e.g. by the monodromy groupoid of the foliation. We will prove the following result:

Theorem 5.1 *For any action of an integrable Lie algebroid $\mathfrak{g}$ on a foliation $\mathcal{F}$, the semi-direct product $\mathfrak{g} \ltimes \mathcal{F}$ is integrable.*

PROOF. Let G be a Lie groupoid which integrates $\mathfrak{g}$. First, let us spell out what it means for the Lie algebroid $\mathfrak{g}$ of G to act on a foliation $\mathcal{F}$ of a manifold N. First, we have a surjective submersion $q : N \to G_0$, and the leaves of $\mathcal{F}$ are contained in the fibers of q. Thus, any fiber $q^{-1}(x)$ is itself a foliated manifold. Next, there is a $C^\infty(G_0)$-linear Lie algebra map $R : \Gamma\mathfrak{g} \to L(N, \mathcal{F})$ into the projectable vector fields on $(N, \mathcal{F})$. In particular, any $X \in \Gamma\mathfrak{g}$ induces a derivation ∇_X on the vector fields Y on N which are tangent to $\mathcal{F}$ by $\nabla_X(Y) = [R(X), Y]$. Finally, this map satisfies the condition that $R(X)$ is projectable along q to the anchor $\rho(X)$, i.e. $dq((R(X)_y) = \rho(X)_{q(y)}$ for any $y \in N$.

Now consider the principal G-bundle $P = N \times_{G_0} G_1$ over N. Here P is the pull-back of $\beta : G_1 \to G_0$ along $q : N \to G_0$, with the evident right G-action $(y, g)g' = (y, gg')$. Consider on P the foliation $\mathcal{G}$ whose tangent space $\mathcal{G}_{(y,g)} \subset T_{(y,g)}(P)$ consists of pairs

$$(R(X)_y + Y_y, X_{\beta(g)}g) \,,$$

where $X \in \Gamma\mathfrak{g}$ and Y is a vector field on N tangent to the foliation $\mathcal{F}$. In other words, the sections of $\mathcal{G}$ are spanned as a $C^\infty(P)$-module by sections of the form $(R(X) + Y, X)$. Here we denote the invariant vector field on G_1 associated to X again by X. The subbundle $\mathcal{G}$ is involutive; indeed, using the fact that R preserves the bracket, we have

$$[(R(X) + Y, X), (R(X') + Y', X')] = (R([X, X']) + Z, [X, X']) \,, \tag{1}$$

where $Z = \nabla_X(Y') - \nabla_{X'}(Y) + [Y, Y']$ is again tangent to the foliation $\mathcal{F}$.

The leaves of this foliation $\mathcal{G}$ are contained in the fibers of the map $\alpha \circ pr_2 : P \to G_0$ which is a part of the principal bundle structure, and the G-action maps leaves into leaves. Thus G also acts on the Lie groupoid $\mathrm{Mon}(P, \mathcal{G})$. By Lemma 2.1, the quotient $K = \mathrm{Mon}(P, \mathcal{G})/G$ is a Lie groupoid over N. We claim that K integrates the semi-direct product $\mathfrak{g} \ltimes \mathcal{F}$. Indeed, the map

$$\Phi : \mathcal{G} \longrightarrow \mathfrak{g} \ltimes \mathcal{F} \,,$$

sending a $\mathcal{G}$-tangent vector field $(R(X) + Y, X)$ to $q^*X \oplus Y$, preserves the bracket (compare with the Equation (1)). Note also that Φ respects the anchor because

$$d(pr_1)(R(X)_y + Y_y, X_{\beta(g)}g) = R(X)_y + Y_y = \rho(q^*X \oplus Y)_y \,,$$

so it is a morphism of Lie algebroids over $\phi = pr_1 : P \to N$. This map Φ induces an isomorphism of the fibers $\mathcal{G}_{(y,g)} \cong (\mathfrak{g} \ltimes \mathcal{F})_y$. Since the α-fibers of the quotient K are the same as those of $\mathrm{Mon}(P, \mathcal{G})$, the vector space $\mathcal{G}_{(y,g)}$ is isomorphic to the fiber $\mathfrak{k}_y$ of the Lie algebroid $\mathfrak{k}$ of K, hence Φ induces an isomorphism $\mathfrak{k} \to \mathfrak{g} \ltimes \mathcal{F}$. $\square$

Example 5.2 If $\mathfrak{g}$ is the Lie algebra of a Lie group, the algebroid associated to an infinitesimal action of $\mathfrak{g}$ on a manifold (Example 1.1 (vi)) is always integrable. This result is due to Dazord [6] (see also [22]), and is a special case of the previous theorem.

5.2 Infinitesimal actions along proper maps

Consider an infinitesimal action of a Lie algebroid $\mathfrak{g}$ over G_0 on a manifold N (viewed as the trivial Lie algebroid over N). Theorem 5.1 implies that the semi-direct product $\mathfrak{g} \ltimes N$ is integrable by some Lie groupoid whenever $\mathfrak{g}$ is integrable. However, if G integrates $\mathfrak{g}$, one can in general not integrate the infinitesimal action to an action of G on N. In this section, we will first show that such an action can be integrated in the special case where the map $q : N \to G_0$ (which is a part of the infinitesimal action) is proper.

Theorem 5.3 *Let G be a source-simply connected Lie groupoid and suppose that the Lie algebroid $\mathfrak{g}$ of G acts on a manifold N along a proper map $q : N \to G_0$. Then there exists an action of G on N along q which integrates the infinitesimal action in the sense that $\mathfrak{g} \ltimes N$ is isomorphic to the Lie algebroid of the translation groupoid $G \ltimes N$.*

PROOF. We view N as a manifold foliated by points, and consider (as in the proof of Theorem 5.1) the foliation $\mathcal{G}$ on $N \times_{G_0} G_1$, whose tangent space $\mathcal{G}_{(y,g)} \subset T_{(y,g)}(N \times_{G_0} G_1)$ consists of pairs $((R(X)_y, X_{\beta(g)}g)$ for $X \in \Gamma\mathfrak{g}$. Also, we consider the foliation $\mathcal{F}$ of G_1 by the α-fibers.

It is clear that the projection $\pi = pr_2 : N \times_{G_0} G_1 \to G_1$ maps $\mathcal{G}$ to $\mathcal{F}$ and restricts to a local diffeomorphism from any leaf L of $\mathcal{G}$ to a leaf L' of $\mathcal{F}$. The projection π is also proper, because it is a pull-back of q which is proper. We will show that this implies that $\pi(L) = L'$ and that $\pi|_L : L \to L'$ is a covering projection. To see this, take any arrow $g \in L'$ and choose vector fields $X_1, \ldots, X_k$ on G_1 such that their values at g form a basis of $\mathcal{F}_g$ (in fact, we can choose $X_1, \ldots, X_k$ to be in $\mathfrak{X}^\alpha_{inv}(G)$). Suppose that U is a small open neighbourhood of g in L' and $\varepsilon > 0$ such that the local flow $\varphi^i_t : U \to L'$ of X_i is well-defined for any $t \in (-\varepsilon, \varepsilon)$, $i = 1, \ldots, k$. We can choose ε so small that the local flows give us an open embedding $\psi : (-\varepsilon, \varepsilon)^k \to L'$ by

$$\psi(t_1, \ldots, t_k) = (\varphi^1_{t_1} \circ \ldots \circ \varphi^k_{t_k})(g) \ .$$

Let $\tilde{X}_i$ be the unique vector field on $N \times_{G_0} G_1$ tangent to $\mathcal{G}$ which projects to X_i along π. Since q is proper we can take U and ϵ so small that the local flow $\tilde{\varphi}^i_t : \pi^{-1}(U) \to \pi^{-1}(L')$ of $\tilde{X}_i$ is well-defined for any $t \in (-\epsilon, \epsilon)$, $i = 1, \ldots, k$. Note that $\pi \circ \tilde{\varphi}^i_t = \varphi^i_t \circ \pi$ because $\tilde{X}_i$ is a lift of X_i. In particular, the map $\tilde{\psi} : (-\varepsilon, \varepsilon)^k \times \pi^{-1}(g) \to \pi^{-1}(L')$ given by

$$\tilde{\psi}(t_1, \ldots, t_k, (y, g)) = (\tilde{\varphi}^1_{t_1} \circ \ldots \circ \tilde{\varphi}^k_{t_k})(y, g)$$

is well-defined and satisfies $\pi \circ \tilde{\psi} = \psi \circ pr_1$. It clearly follows that $\tilde{\psi}$ is an open embedding. The map $\tilde{\psi}$ also maps the product foliation of $(-\varepsilon, \varepsilon)^k \times \pi^{-1}(g)$ to $\mathcal{G}$. In particular, the projection $\pi|_L : L \to L'$ is a covering.

Now recall that in the case at hand L' is an α-fiber of G and hence simply connected. It follows that the covering projection $\pi|_L : L \to L'$ is in fact a

diffeomorphism. We can now define a G-action on N

$$\vartheta : G_1 \times_{G_0} N \longrightarrow N$$

along q as follows: for any arrow g of G and any point y of N satisfying $\alpha(g) = q(y)$, let gy be the unique point of N such that (gy, g) lies on the same leaf of $\mathcal{G}$ as $(y, 1_{q(y)})$. It is clear that the map ϑ satisfies the identities for an action. To see that ϑ is smooth, observe that a lift of a holonomy extension of a path γ in a leaf of $\mathcal{F}$ can be obtained as a holonomy extension of a lift of γ. It is straightforward to check that this G-action on N indeed integrates the infinitesimal action of $\mathfrak{g}$ on N. $\square$

REMARK. Note that the assumption that q is proper can be replaced by the assumption that all the vector fields $R(X)$ are complete.

Corollary 5.4 *Let G and H be source-simply connected Lie groupoids with Lie algebroids $\mathfrak{g}$ respectively $\mathfrak{h}$, and suppose that $\mathfrak{g}$ acts on $\mathfrak{h}$ along a proper map $q : H_0 \to G_0$. Then there exists an action of G on H which integrates the infinitesimal action, in the sense that the semi-direct product $\mathfrak{g} \ltimes \mathfrak{h}$ is isomorphic to the Lie algebroid of the semi-direct product groupoid $G \ltimes H$.*

PROOF. Consider the action of G on H_0 given by Theorem 5.3. The Lie algebroid $\mathfrak{h}$ is a family of Lie algebroids over G_0, so its pull-back along $pr_2 : G_1 \times_{G_0} H_0 \to H_0$ is again a Lie algebroid. This pull-back is equipped with a foliation whose leaves are the fibers of pr_2. These fibers are isomorphic to the α-fibers of G. The action of $\mathfrak{g}$ on $\mathfrak{h}$ defines a flat partial connection on $pr_2^* \mathfrak{h}$ along the leaves of the foliation of $G_1 \times_{G_0} H_0$. Since the leaves are simply connected, there is a well-defined transport along the leaves, which defines an action of G on the algebroid $\mathfrak{h}$. This action can now be integrated by Theorem 3.6. Further details are straightforward. $\square$

Let $\mathfrak{g}$ be a Lie algebroid over M, and let $\mathfrak{h}$ be a bundle of Lie algebras over M (Example 1.1 (iv)). An *extension* $\mathfrak{k}$ of $\mathfrak{g}$ by $\mathfrak{h}$ is an exact sequence of Lie algebroids over M of the form

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{k} \longrightarrow \mathfrak{g} \longrightarrow 0 \ .$$

It is called *split* if it splits by a morphism of Lie algebroids $\mathfrak{g} \to \mathfrak{k}$ over M.

Corollary 5.5 *Any split extension of an integrable Lie algebroid over M is integrable.*

PROOF. A split extension as above represents $\mathfrak{k}$ as the semi-direct product $\mathfrak{g} \ltimes \mathfrak{h}$ for an action by $\mathfrak{g}$ on $\mathfrak{h}$ along the identity $M \to M$. Thus the result follows from the previous corollary and integrability of $\mathfrak{h}$ (Example 1.4 (iii)). $\square$

5.3 Infinitesimal actions on the Lie algebroids of source-compact groupoids

We consider again two Lie groupoids G and H with Lie algebroids $\mathfrak{g}$ and $\mathfrak{h}$, respectively, and an infinitesimal action of $\mathfrak{g}$ on $\mathfrak{h}$ along a map $q : H_0 \to G_0$. In this section we shall prove that the semi-direct product $\mathfrak{g} \ltimes \mathfrak{h}$ is integrable in the case where H is a source-compact source-simply connected groupoid.

Definition 5.6 *A Lie groupoid H is called source-compact if it is Hausdorff and if the source map $\alpha : H_1 \to H_0$ is proper.*

REMARK. Note that this implies that the other structure maps β, *uni* and μ of H are also proper. Also note that if H is proper then so is the source-connected subgroupoid $H^{(0)}$ of H.

Theorem 5.7 *If $\mathfrak{g}$ and $\mathfrak{h}$ are integrable Lie algebroids and if $\mathfrak{h}$ has a source-simply connected source-compact integral, then any semi-direct product $\mathfrak{g} \ltimes \mathfrak{h}$ is integrable.*

PROOF. Let G be a source-simply connected integral of $\mathfrak{g}$ and H a source-simply connected source-compact integral of $\mathfrak{h}$. Let ∇ be an action of $\mathfrak{g}$ on $\mathfrak{h}$ along $q : H_0 \to G_0$ over R. For any $X \in \Gamma\mathfrak{g}$ let $w(X)$ be the multiplicative vector field on H with $\mathcal{L}_{w(X)} = \nabla_X$ and $w(X)|_{H_0} = R(X)$ (Theorem 4.5). In particular, the map $w : \Gamma\mathfrak{g} \to \mathfrak{X}^\mu(H)$ is a $C^\infty(G_0)$-linear homomorphism of Lie algebras, hence it induces a bundle map $\bar{w} : (q \circ \beta)^*\mathfrak{g} \to T(H_1)$ by $\bar{w}(h, u) = w(X)_h$, for any $X \in \Gamma\mathfrak{g}$ with $X_{\pi(u)} = u$.

Now define a foliation $\mathcal{F}$ of the manifold

$$M = H_1 \times_{G_0} G_1 = \{(h, g) \in H_1 \times G_1 \,|\, q(\beta(h)) = \beta(g)\}$$

by

$$\mathcal{F}_{(h,g)} = \{(Y_{\beta(h)}h + w(X)_h, X_{\beta(g)}g) \,|\, X \in \Gamma\mathfrak{g}, \, Y \in \Gamma\mathfrak{h}\} \subset T_{(h,g)}M \;.$$

Note that $(Y_{\beta(h)}h + w(X)_h, X_{\beta(g)}g)$ is indeed tangent to M because

$$
\begin{aligned}
d(q \circ \beta)(Y_{\beta(h)}h + w(X)_h) &= dq(d\beta(w(X)_h)) = dq((R(X))_{\beta(h)}) \\
&= \rho(X)_{q(\beta(h))} = \rho(X)_{\beta(g)} = d\beta(X_{\beta(g)}g) \;.
\end{aligned}
$$

The fact that w is a $C^\infty(G_0)$-linear homomorphism implies that $\mathcal{F}$ is a sub-bundle of TM, with $\dim(\mathcal{F}) = \operatorname{rank}(\mathfrak{g}) + \operatorname{rank}(\mathfrak{h})$. We have to show that $\mathcal{F}$ is involutive. For any $X, X' \in \Gamma\mathfrak{g}$ and $Y, Y' \in \Gamma\mathfrak{h}$ we have

$$
\begin{aligned}
&[(Y + w(X), X), (Y' + w(X'), X')] \\
&= \; ([Y + w(X), Y' + w(X')], [X, X']) \\
&= \; ([Y, Y'] + [w(X), Y'] + [Y, w(X')] + [w(X), w(X')], [X, X']) \\
&= \; ([Y, Y'] + \nabla_X(Y') - \nabla_{X'}(Y) + w([X, X']), [X, X']) \;, \qquad (2)
\end{aligned}
$$

and this is again a section of $\mathcal{F}$ since $[Y, Y'] + \nabla_X(Y') - \nabla_{X'}(Y) \in \Gamma\mathfrak{h}$. Here we denoted the invariant vector field on G_1 corresponding to X again by X, and the same for X', Y and Y'.

When we view the foliation $\mathcal{F}$ of M as a Lie algebroid over M, there is a map of Lie algebroids

$$\Phi : \mathcal{F} \longrightarrow \mathfrak{g} \ltimes \mathfrak{h}$$

over the map $\beta \circ pr_1$, sending an $\mathcal{F}$-tangent vector field $(Y + w(X), X)$ to $q^*X \oplus Y$. Indeed, the map Φ clearly preserves the anchor, and it preserves the bracket by Equation (2). Note, in addition, that Φ restricts to an isomorphism on each fiber. The algebroid $\mathcal{F}$ is integrable by $\operatorname{Mon}(M, \mathcal{F})$.

The manifold M comes equipped with a natural right H-action along the map $\alpha \circ pr_1$, which is principal with respect to the projection $\beta \times id : M \to H_0 \times_{G_0} G_1$. Also, M has a natural right G-action along $\alpha \circ pr_2$, which is principal with respect to the projection $pr_1 : M \to H_1$. These two actions commute with each other, and together they make M into a principal $(H \times G)$-bundle over H_0. Note that the foliation $\mathcal{F}$ is tangent to the fibers of the map $\alpha \circ pr_2$, so the G-action lifts to a G-action on the monodromy groupoid $\text{Mon}(M, \mathcal{F})$. On the other hand, the foliation $\mathcal{F}$ is in general not tangent to the fibers of $\alpha \circ pr_1$. Despite this we will show that the H-action on M may be lifted to an H-action on $\text{Mon}(M, \mathcal{F})_1$. However, this is not an action of H on the groupoid in the sense of Subsection 2.1. We will show that $\text{Mon}(M, \mathcal{F})$ can be factored by G and H with respect to these two actions to give a Lie groupoid K over H_0 with the same α-fibers as $\text{Mon}(M, \mathcal{F})$. It will then be clear from the construction and the properties of the map Φ that K integrates $\mathfrak{g} \ltimes \mathfrak{h}$.

To describe the H-action, recall that source-connectedness of H implies that $q \circ \alpha = q \circ \beta$, so H is a family of Lie groupoids over G_0. We can take the pull-back of this family along the target map $\beta : G_1 \to G_0$ to get a family of groupoids over G_1, as in the following diagram:

$$
H_1 \times_{H_0} H_1 \times_{G_0} G_1 \xrightarrow{\mu \times id} H_1 \times_{G_0} G_1 \overset{\overset{uni \times id}{\overset{\curvearrowleft}{\underset{\beta \times id}{\overset{\alpha \times id}{\rightrightarrows}}}}}{} H_0 \times_{G_0} G_1 \xrightarrow{pr_2} G_1 \qquad (3)
$$

The first pull-back here consists of $(h, h', g) \in H_1 \times H_1 \times G_1$ satisfying $\alpha(h) = \beta(h')$ and $\beta(h) = \beta(g)$.

Consider now the foliation $\mathcal{G}$ on G_1 given by α-fibers. The action of $\mathfrak{g}$ on H_0 defines a foliation $\mathcal{G}^{(0)}$ of $H_0 \times_{G_0} G_1$ given by

$$
\mathcal{G}^{(0)}_{(y,g)} = \{(R(X)_y, X_{\beta(g)}g) \mid X \in \Gamma\mathfrak{g}\} \, .
$$

Similarly, define a foliation $\mathcal{G}^{(1)}$ of $H_1 \times_{G_0} G_1$ by

$$
\mathcal{G}^{(1)}_{(h,g)} = \{(w(X)_h, X_{\beta(g)}g) \mid X \in \Gamma\mathfrak{g}\} \, ,
$$

and a foliation $\mathcal{G}^{(2)}$ of $H_1 \times_{H_0} H_1 \times_{G_0} G_1$ by

$$
\mathcal{G}^{(2)}_{(h,h',g)} = \{(w(X)_h, w(X)_{h'}, X_{\beta(g)}g) \mid X \in \Gamma\mathfrak{g}\} \, .
$$

Observe that, since each $w(X)$ is a multiplicative vector field on H, all the maps in Diagram (3) map leaves to leaves. Now note that the assumption that H is source-compact implies that the maps $\alpha \times id$, $\beta \times id$ and $\mu \times id$ of Diagram (3) are proper. By the argument given in the proof of Theorem 5.3 it follows that each leaf of $\mathcal{G}^{(1)}$ projects along $\beta \times id$ (and also along $\alpha \times id$) onto a leaf of $\mathcal{G}^{(0)}$ as a covering projection. The same is true for the leaves of $\mathcal{G}^{(2)}$ with respect to the projection $\mu \times id$. Note also that $\mathcal{G}^{(1)}$ is a subfoliation of $\mathcal{F}$. There is another foliation $\mathcal{K}$ of $H_1 \times_{H_0} H_1 \times_{G_0} G_1$ given by

$$
\mathcal{K}_{(h,h',g)} = \{(w(X)_h + Y_{\beta(h)}h, w(X)_{h'}, X_{\beta(g)}g) \mid X \in \Gamma\mathfrak{g}, Y \in \Gamma\mathfrak{h}\} \, .
$$

The map $\mu \times id$ maps the leaves of $\mathcal{K}$ onto the leaves of $\mathcal{F}$ as a covering projection.

Now take any path γ in a leaf of $\mathcal{F}$ from (h_0, g_0) to (h_1, g_1). Observe first that $\bar{\gamma} = (\alpha \times id) \circ \gamma$ is a path tangent to the foliation $\mathcal{G}^{(0)}$. Hence for any $h \in H(\,\text{-}\,, \alpha(h_0))$ there exists a unique lift $\tilde{\gamma}$ of $\bar{\gamma}$ along $\beta \times id$ tangent to $\mathcal{G}^{(1)}$ with $\tilde{\gamma}(0) = (h, g_0)$. Put $\delta = pr_1 \circ \tilde{\gamma}$. Now we may use the H-action on M to define a new path γh in M by

$$(\gamma h)(t) = \gamma(t) \delta(t) \ .$$

The fact that $\mu \times id$ maps $\mathcal{K}$ to $\mathcal{F}$ implies that γh is again a path in a leaf of $\mathcal{F}$. Note that $(\beta \times id) \circ (\gamma h) = (\beta \times id) \circ \gamma$. It h' is another arrow in $H(\,\text{-}\,, \alpha(h))$, we lift $\bar{\gamma}$ to $\tilde{\gamma}$ along $\beta \times id$ as before, and we lift $\overline{\gamma h}$ along $\beta \times id$ to $\tilde{\gamma}'$ tangent to $\mathcal{G}^{(1)}$ with $\tilde{\gamma}'(0) = (h', g_0)$. These give us $\delta = pr_1 \circ \tilde{\gamma}$ and $\delta' = pr_1 \circ \tilde{\gamma}'$. It follows that $(\delta, \delta', pr_2 \circ \gamma)$ is a path in a leaf of $\mathcal{G}^{(2)}$, and its projection along $\mu \times id$ is a path in a leaf of $\mathcal{G}^{(1)}$ which lifts $\bar{\gamma}$ with value (hh', g_0) at $t = 0$. Therefore

$$\gamma(hh') = (\gamma h)h' \ .$$

A similar argument shows that

$$(\gamma' \gamma)h = (\gamma' \tilde{h})(\gamma h) \ , \tag{4}$$

where $\tilde{h}$ is now the unique arrow in H satisfying $(\gamma h)(1) = \gamma(1)\tilde{h}$ (or $\tilde{h} = \delta(1)$ for δ as above).

Finally, for any arrow $\sigma \in \mathrm{Mon}(M, \mathcal{F})_1$ we may define

$$\sigma h \in \mathrm{Mon}(M, \mathcal{F})_1$$

by $\sigma h = [\gamma h]$, where γ is any path representing σ, i.e. $\sigma = [\gamma]$. This is well-defined since the definition is given by path-lifting along a covering projection. The properties mentioned above imply that this defines an H-action on $\mathrm{Mon}(M, \mathcal{F})_1$ along the map $\epsilon = \alpha \circ pr_1 \circ \alpha : \mathrm{Mon}(M, \mathcal{F}) \to H_0$. This action commutes with the G-action, and we may take the quotient $K = \mathrm{Mon}(M, \mathcal{F})/G/H$, which is a smooth manifold because it can be identified with the pull-back of the source map of $\mathrm{Mon}(M, \mathcal{F})$ along $H_0 \to M$. Using Equation (4) it is easy to check that K is a Lie groupoid over H_0. Now Φ induces an isomorphism from the Lie algebroid of K to $\mathfrak{g} \ltimes \mathfrak{h}$ over H_0. $\square$

REMARK. Note that the assumption that H is source-compact can be replaced by the assumption that H is Hausdorff and that all the vector fields $w(X)$ are complete.

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